Survey of Laser Models (with a focus on Maxwell-Bloch equations)

Ainsley Lai, Wenyu Jiao, Xi Liang

March 2022

Abstract

With the rising usage of lasers across a variety of domains, it is essential to have a better understanding of how they work. This paper will analyze mathematical models of lasers derived from Haken, Milonni and Eberly, and Maxwell-Bloch. We will study the formulation of the models by studying the physics of the laser and quantum electrodynamics as well as their long-term behavior when tuning different parameters. Through analysis, we've found that, for the most part, the 2-D and 3-D model both behave intuitively with differing results when pumping is absent. Through observing the models, we will find ways to improve our analysis and possible improvements for the models.

1 Introduction

The invention of the laser is inseparable from the accumulation of long-term basic research and technological progress in physics, especially concerning electromagnetism. Figures from Planck[3] to Maiman and Haken[6] have contributed numerous findings on the topic to allow practical applications of lasers.

The fields of laser application mainly include six fields: industry, medical treatment, commerce, scientific research, information, and military. Industrial applications include material processing and measurement control; medical applications include therapy and diagnosis; and commercial applications. Another interesting thing is the application of lasers in biology.

In June 2014, American scientists developed a laser system, called the Fly Mind Altering Device (FlyMAD), that can quickly control the thoughts of fruit flies. The system uses a camera to track a fly's movement and by firing a specially tuned laser at the fly, can trigger photothermal-related neural pathways in the fruit fly's brain. This application provides a way to study light-controlled thinking.[4]

Usage of lasers is also found in optical tweezers, in which a laser is used to hold/move a microscopic object, hence the name tweezers. Due to their ability to manipulate microscopic objects, such as cells atoms, they are a useful tool in biology, nanochemistry, and quantum optics. [1]

The focus of this paper will be on three models presented by Strogatz in *Nonlinear Dynamics and Chaos* [5]; a 1-D, 2-D, and 3-D Maxwell-Bloch ODE model of lasers, each increasing in sophistication. As the title of the source would suggest, this is a nonlinear system and in the case of the Maxwell-Bloch model, a Lorenz system that presents the problem of chaotic behavior (which many lasers do not exhibit). The first model, a 1-D system, references Haken's derivation[6] in which the laser model is seen through the lens of photon number. In this model, Haken reduces to a 1-D system by assuming that the ground level is kept empty and simplifies the expression for excited atoms in terms of photons. The 2-D model that Strogatz[5] presents from Milonni and Eberly[7], in which the dynamics of excited atoms are expressed, is an extension of Haken's simple model. Lastly, we review Strogatz' 3-D model based on the Maxwell-Bloch equations. This particular model views the system in a more specific manner than the quantity of photons and excited atoms. However, this model is similar to a Lorenz equation and thus has chaotic behavior. Thus, we have three different models, from simple to complex, to work with. While the simpler models are less accurate, they may be less prone to chaotic behavior.

It goes without saying that Strogatz is not the only one behind these models. The culmination of knowledge from multiple sources like Haken, Milonni, and Weiss are what allows us to use the models we have now and this paper will be pulling from information in their publications (references at the end of the paper). Formulation of these models will be given in the next section.

To further study the presented laser models, we will examine the fixed point behavior of the systems, the threshold/bifurcation of the laser, and the general behavior of these models.

2 Formulation

2.1 1-D Model

2.1.1 Physical System

Beginning with the simplest of our three models, Strogatz examines one type of laser: a solid-state laser [5]. In this system, laser-active atoms are pumped into a solid-state matrix in which the laser will behave as a lamp when the pumping is weak. Upon reaching a certain pumping threshold, the atoms will start to oscillate in phase and cohere: turning into a laser.

2.1.2 ODE System

Strogatz then presents the simplified model by Haken [6] that models the system of n, the number of photons, using a 1-D ODE describing the gain and loss of photons in the laser field: $GnN - kn$ (gain - loss) where N is the number of excited atoms.

$$\dot{n} = Gn\left(N_0 - \alpha n\right) - kn$$
$$N(t) = N_0 - \alpha n$$

Gain is derived from the stimulated emission where photons stimulate excited atoms to produce more photons and happens at a positive nonzero rate, G, proportional to both. The loss term, $-kn$, is told to be the escape of photons from the system which intuitively occurs at a positive nonzero rate proportional to the photons.

Interestingly, Strogatz explains that the N can be written as an expression of photons, thereby reducing the system to 1-D when it would ordinarily be 2-D. This brings in the assumption that in the absence of laser action, the excited atoms will be held at a constant by the pump. Thus, laser action would introduce a decreasing linear term where excited atoms decrease by some rate, α, as they emit a photon and drop down to a lower energy level. This model would mean that the pump does not continuously pump excited atoms into the system or that the pump shuts off as laser action occurs. In that case, the laser would be short-lived and be an inaccurate example of a man-made laser tool as it would become prone to possibly decaying below the laser threshold.

2.1.3 Law of Mass Action

We may also observe the system of laser photons, n, and excited atoms, N, through the law of mass action:

$$n + N \xrightarrow{G} 2n \qquad\qquad n \xrightarrow{k} 0$$

In this law of mass action representation, we can see the gain and loss reactions more clearly. If we consider the ODE of N, one would get $\dot{N} = -GnN$, signifying a decreasing N, which is in line with the earlier idea in which the quantity of excited atoms only decreases in this model.

2.1.4 Assumptions and Simplifications

In this section, let us recount the assumptions and simplifications present in the model. Recall that this model was taken from Haken and is simplified to only consider the 'essential physics' of the system and represent it using a dynamical variable of photon quantity and writing it in terms of gain and loss. This may ignore the more specific traits inherent to photons (such as the electromagnetic field and its interactions with the system).

$$N(t) = N_0 - \alpha n$$

Another assumption that is made is Haken's technique of reducing the system by writing N in terms of n. This is done by assuming that the number of excited atoms, N, will be constant until the laser process begins and reduces N without any source of replenishment.

2.2 2-D Model

2.2.1 ODE System

For the same solid-state laser system, Strogatz also provides another laser model from Milonni and Eberly [7] (who take their own approximations) that acts as an extension of the previous solid-state laser.

$$\dot{n} = GnN - kn$$
$$\dot{N} = -GnN - fN + p$$

Rather than write the number of excited atoms in terms of photons, quantum mechanics[7] gives us a 2-D ODE model where N has its own dynamics. The dynamics of N is described as decreasing by the amount of photon gain (GnN) and the natural relaxation of excited atoms (fN). The relaxation of excited atoms results in out-of-phase photons and thus does not contribute to the production of n. In this model, the only increase in N comes from the pumping parameter, denoted as p. This model seems to better describe the decay of N as well as include a constant pumping parameter. For this model, coefficients G and f are set to be positive since the decay is characterized by the negatives. Parameter p can have either sign, positive or negative, representing a constant change in excited atoms.

2.2.2 Law of Mass Action

Again, we may consider the model using the law of mass action:

$$n + N \xrightarrow{G} 2n \qquad\qquad n \xrightarrow{k} 0$$
$$N \xrightarrow{f} 0$$

By doing this, we can see where all but one one of the terms come from. These reactions model stimulated emission and natural decay rates. The p parameter is an external pumping of excited atoms and is thus a lone constant unassociated with n or N.

2.2.3 Assumptions and Simplifications

While this model does improve on the previous by providing a more accurate expression for the dynamics of N in regards to decay and growth, it does bring in the assumption that the pumping parameter is constant and non-variable. This may be inaccurate for practical laser usage where excited atoms may be pumped into the system at varying rates. This model still uses a model based on n and N though and may be prone to ignore the more specific traits of the electromagnetic field and such.

2.3 3-D Model

2.3.1 ODE System

Using Maxwell-Bloch equations, we can get a more specific model of the laser system. This new model is a first order 3-D ODE system of E, P, and D (electric field, atom polarization, and population inversion).

$$\dot{E} = \kappa(P - E)$$
$$\dot{P} = \gamma_1(ED - P)$$
$$\dot{D} = \gamma_2(\lambda + 1 - D - \lambda EP)$$

We can find that E relates to the number of photons due to light being an electromagnetic wave. To be more specific, E is an electric field in a single longitudinal cavity mode. This ODE would become nonlinear when oscillating between two discrete energy levels. We also find that P and D are related to the excited atoms, measuring certain properties of the atoms. As stated by Strogatz[5], κ is the laser cavity's decay rate due to beam transmission, γ_1 and γ_2 are decay rates of atomic polarization and population inversion, respectively, and λ is the pumping energy parameter.

To go into further detail of the derivation of these models and their form, we can draw analogies from a Lorenz model of fluid instabilities and use a rotating wave approximation[9]. By applying them to equations for field strength, E, polarization, P, and inversion, D, we obtain our ODE system. By doing so, one can find that the 3-D ODE system may be equivalent to a Lorenz model used to describe fluid instabilities.

Working in reverse, a similarity between the 2-D and 3-D models can be made by setting $n = cE^2$ where c is some constant in the fast P dynamics where $P = ED$ or $\dot{P} = 0$:

$$
\begin{aligned}
E &= c\sqrt{n} \\
\dot{E} &= k\left(c\sqrt{n}D - c\sqrt{n}\right) \\
&= \overbrace{kc\sqrt{n}D}^{GnN} - \overbrace{ck\sqrt{n}}^{-kn} \\
\dot{D} &= \overbrace{\gamma_2\lambda + \gamma_2}^{p} - \overbrace{\gamma_2 D}^{fN} - \overbrace{\gamma_2\lambda c^2 nD}^{GnN}
\end{aligned}
$$

As denoted by the overbraces, it is possible that some of the terms correspond with those in the 2-D model. This hypothesis is more apparent when we consider the meanings of E and D and their associations. As mentioned before, E, is related to photons, n, via the electromagnetic field/E-M wave whereas D, atom inversion, is related to the excited atoms, N.

3 Analysis

3.1 1-D Model

Now we are looking at the 1-D model of the laser. This model is inspired by Haken[6]. This model is a simplified model built in basic physics. $n(t)$ as a dynamic variable is the number of photons in the laser field. This change is the gain minus the loss. Because this process occurs through random encounters of photons and excited atoms, it is $N(t)$. $N(t)$ will decrease as photons are emitted. Suppose N_0 is the number of excited atoms that the pump maintains, and the actual number of excited atoms will decrease during lasing.

$$
\begin{aligned}
\dot{n} &= GnN - kn \\
N(t) &= N_0 - \alpha n
\end{aligned}
$$

Here α greater than 0 is the rate of falling into the ground state. So we can get :

$$
\begin{aligned}
\dot{n} &= Gn(N_0 - \alpha n) - kn \\
\dot{n} &= (GN_0 - k)n - (\alpha G)n^2
\end{aligned}
$$

To get the fix point, $\dot{n}$ should be 0. which we can have:

$$\dot{n} = Gn(N_0 - \alpha n) - kn = 0$$

The answer of this equation is either n $= 0$ or n $= \frac{GN_0-k}{\alpha G}$. So, we can have the value of N_0 which is $\frac{k}{G}$ by having n $= \frac{GN_0-k}{\alpha G} = 0$. In conclusion, the fix point will be at either n $= 0$, or, n $= \frac{GN_0-k}{\alpha G}$.

$$n = (GN_0 - k)n - (\alpha G)n^2$$
$$\dot{n} = GN_0 - k - 2\alpha Gn$$

With the fix points we have above, we can get the equations like :

$$\dot{n}(0) = GN_0 - k$$
$$\dot{n}(\frac{GN_0 - k}{\alpha G}) = -GN_0 + k$$

3.1.1 Case 1 & 2

From the above equations, we can see the derivative function having 0 and $\frac{GN_0-k}{\alpha G}$ as input is similar. They all have (GN_0-k) with just the sign being different. So, let's observe the possible cases for GN_0-k. The first case is when GN_0-k is smaller than 0, which means $N_0 < \frac{k}{G}$. This means derivative n(0) is negative and smaller then 0 which is a stable point, while derivative n($\frac{GN_0-k}{\alpha G}$) is positive and larger than 0 which is unstable. We know that n is the quality of photon which cannot be negative. So, there is only one point n $= 0$ existing, and that point is a stable point. The second case is when GN_0-k is equal to 0. This is the situation when $N_0 = \frac{k}{G}$. Now if we put n $= 0$, and n $= \frac{GN_0-k}{\alpha G}$ to the derivative equation, we will have both answers being 0. This means the point is a semi-stable point. There is still one point fitting in which is n $= 0$ since the two fix point is same.

3.1.2 Case 3

The third case is a different one, because in this case n is a positive number which means there will be like 2 points. For this one, we are setting GN_0-k to be larger then 0 which is $N_0 > \frac{K}{G}$. Also by using the derivative n(0) and derivative n($\frac{GN_0-k}{\alpha G}$), we will get 2 points this time. Derivative n(0) is positive so the point is unstable while derivative n($\frac{GN_0-k}{\alpha G}$) is negative and stable. In this situation, there is two point that one is stable while the other is unstable.

3.2 2-D Model

Let us consider and analyze the 2-D model from Milonni and Eberly[7]:

$$\dot{n} = GnN - kn$$
$$\dot{N} = -GnN - fN + p$$

We find that it has the following nullclines

$$\text{n}-\text{nullcline}: N = \frac{k}{G}$$
$$\text{N}-\text{nullcline}: N = \frac{p}{Gn+f}$$
$$\text{Fixed Points } (\text{n}^*, \text{N}^*): \left(\frac{p}{k} - \frac{f}{G}, \frac{k}{G}\right), \left(0, \frac{p}{f}\right)$$

To analyze this, we can consider the Jacobian where $f = \dot{n},\ g = \dot{N}$:

$$\mathbf{J} = \begin{pmatrix} \frac{\partial f}{\partial n} & \frac{\partial f}{\partial N} \\ \frac{\partial g}{\partial n} & \frac{\partial g}{\partial N} \end{pmatrix} = \begin{pmatrix} GN-k & Gn \\ -GN & -Gn-f \end{pmatrix}$$
$$\mathbf{J}_{\left(\frac{p}{k}-\frac{f}{G},\ \frac{k}{G}\right)} = \begin{pmatrix} 0 & \frac{Gp}{k}-f \\ -k & -\frac{Gp}{k} \end{pmatrix}$$
$$\text{Tr}(\mathbf{J}) = -\frac{Gp}{k} \quad D = Gp - fk \quad \Delta = -\frac{Gp}{k} - 4\left(Gp - fk\right)$$
$$\mathbf{J}_{\left(0, \frac{p}{f}\right)} = \begin{pmatrix} \frac{Gp}{f}-k & 0 \\ -\frac{Gp}{f} & -f \end{pmatrix}$$
$$\text{Tr}(\mathbf{J}) = \frac{Gp}{f} - k - f \quad D = fk - Gp \quad \Delta = \frac{Gp}{f} - k - f - 4\left(fk - Gp\right)$$

3.2.1 Case 1 & 2

Let's observe the cases where $p \leq 0$. Given the nature of the parameters, we can see that if p is negative, the fixed point at $\left(\frac{p}{k} - \frac{f}{G}, \frac{k}{G}\right)$ will be a saddle point by observing D and referencing the Poincaré diagram[2]. Also note that it will lead to a fixed point with a coordinate out of quadrant 1 and suggest a negative quantity of photons, which does not exist and points towards a scenario where the values go to the axes/zero. A similar case occurs with $p = 0$. Despite having a zero trace, the determinant D becomes negative, suggesting a saddle point that lies outside of quadrant 1 in the land of negative quantities.

For the fixed point at $\left(0, \frac{p}{f}\right)$, we can find that it'd be a sink when $p \leq 0$. Of course, this would be a negative fixed point. This signifies a scenario where both values will decay due to lack of new excited atoms which in turn decrease stimulated emission, the only term in the model that produces photons. Mathematically, this leads to negative N values (if $p < 0$) but realistically, N would stop at 0 and n would decrease on its own to 0.

3.2.2 Case 3

The case where $p > 0$ is likely the more relevant case for the scenario (at least until we want the pumping to be turned off). This case is where we can directly observe the pumping threshold and how, past a certain pumping parameter, the laser process occurs.

First let us consider the fixed point at $\left(0, \frac{p}{f}\right)$. With positive p, it will be a sink if $p < \frac{fk}{G}$ and a saddle point when $p > \frac{fk}{G}$. This sink case represents the scenario when pumping is not strong enough to start the laser, leading to zero photons and left over excited atoms (constant but weak pumping keeps N above zero). For the laser to occur, the parameter must exceed the threshold, $p > \frac{fk}{G}$. This results in the other fixed point, $\left(\frac{p}{k} - \frac{f}{G}, \frac{k}{G}\right)$, being a nonzero n sink node, meaning that we have enough photons and excited atoms to maintain the laser. Otherwise, this fixed point would be a saddle point with negative n and the sink on $\left(0, \frac{p}{f}\right)$ would be the more physically relevant point if $p < \frac{fk}{G}$. From plotting we can see that $p = \frac{fk}{G}$ would not be able start the laser. Instead, it would act as a sink node for positive n and N.

3.3 3-D Model

The laser model based on the the Maxwell-Bloch equations is a sophisticated model with three differential equations describing the dynamics of the electric field E, the mean polarization P of the atoms, and the population inversion D. We will present the equations of the system and the meaning of parameters, and then we start to analyze the system.

$$\begin{aligned}
\dot{E} &= \kappa(P - E) \\
\dot{P} &= \gamma_1(ED - P) \\
\dot{D} &= \gamma_2(\lambda + 1 - D - \lambda EP)
\end{aligned}$$

where κ is the decay rate in the laser cavity due to beam transmission, γ_1, γ_2 are decay rates of the atomic polarization and population inversion, respectively, and l is a pumping energy parameter. The parameter λ may be positive, negative, or zero; all the other parameters are positive.

3.3.1 Case 1

We first consider the simplest case $\gamma_1 \approx \gamma_2 \gg \kappa$[11]. In this case, P and D relax rapidly to steady values and hence may be adiabatically eliminated.
We assume $\dot{P} \approx 0$ and $\dot{D} \approx 0$, express P and D in terms of E, and thereby derive a first-order equation for E.

$$\dot{P} = \gamma_1(ED - P) = 0 \qquad \dot{D} = \gamma_2(\lambda + l - D - \lambda EP) = 0$$

$$P = ED \qquad P = \frac{\lambda + 1 - D}{\lambda E}, (\lambda E \neq 0)$$

Combining the two equations for P, we have

$$D = \frac{\lambda + 1}{1 + \lambda E^2}, (1 + \lambda E^2 \neq 0).$$

Thus, P can be expressed as

$$P = ED = \frac{(\lambda + 1)E}{1 + \lambda E^2}, (1 + \lambda E^2 \neq 0).$$

Plug in the expression for P back to the 1-st order ODE $\dot{E}$, we have

$$\dot{E} = \kappa(\frac{(\lambda+1)E}{1+\lambda E^2} - E).$$

Finding the fixed points for $\dot{E}$ when $\lambda > 0$,

$$\begin{aligned} \dot{E} = \kappa(\frac{(\lambda+1)E}{1+\lambda E^2} - E) &= 0 \\ \frac{\kappa\lambda E(1-E^2)}{1+\lambda E^2} &= 0 \\ \Rightarrow E = 0, \quad E = -1, \quad E &= 1. \end{aligned}$$

When $\lambda > 0$, there are three fixed points. $E = 0$ is the source, and $E = -1$ and $E = 1$ are the sinks. When $\lambda = 0$, all values of E are fixed points, and they are all stable but not attracting. The behavior of E gets complicated when $\lambda < 0$. If $\lambda = -1$, there is only one fixed point at $E = 0$. If $\lambda < 0$ and $\lambda \neq -1$, there are five fixed points at $E = 0, \pm 1, \pm\sqrt{-\frac{1}{\lambda}}$. $E = 0$ is always a sink, but the types of other fixed points depend on λ. When $\lambda < -1$, $E = \pm 1$ are the sinks, and $E = \pm\sqrt{-\frac{1}{\lambda}}$ are the sources. When $-1 < \lambda < 0$, $E = \pm 1$ are the sources, and $E = \pm\sqrt{-\frac{1}{\lambda}}$ are the sinks.
Example plots with different λ are shown in Figure 1. It is important to note that the trajectories in Figure 1c are not oscillating around 2 or -2, the simulation should stop once the trajectories reach those two points since $\dot{E}$ is undefined when $1 + \lambda E^2 = 0$. The fixed points $E = \pm\sqrt{-\frac{1}{\lambda}}$ are attracting but are not the common sinks we normally encounter in other problems.
The number and the values of the fixed points E^* depend on the parameter λ. To make the relationship between E and λ more straightforward, we have the bifurcation diagram of E^* vs. λ with $\kappa = 1$ in Figure 2.

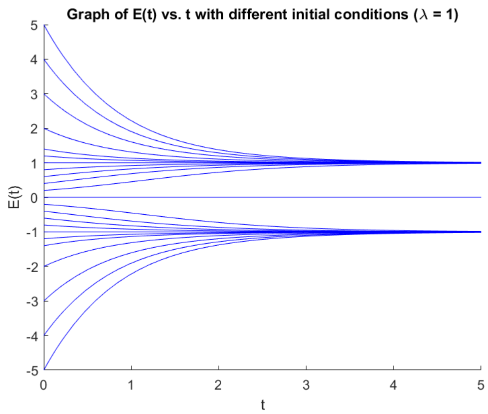
3.3.2 Case 2

We then examine the case $\gamma_1 \gg \kappa \geq \gamma_2$, which is a bit more complex than Case 1. In this case, P relax rapidly to steady value and hence may be adiabatically eliminated. We assume $\dot{P} \approx 0$, so

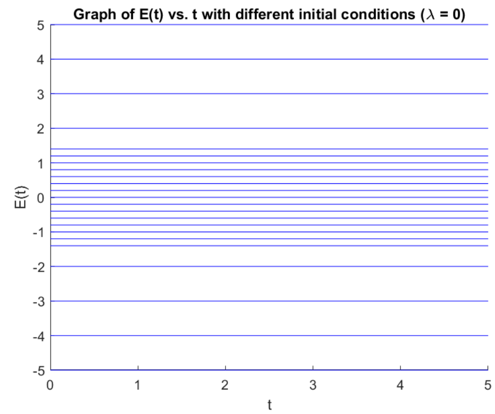
$$\begin{aligned} \dot{P} = \gamma_1(ED - P) &= 0 \\ ED - P &= 0 \\ P &= ED. \end{aligned}$$

Then, we plug the expression for P to $\dot{E}$ and $\dot{D}$ to obtain a 2D system

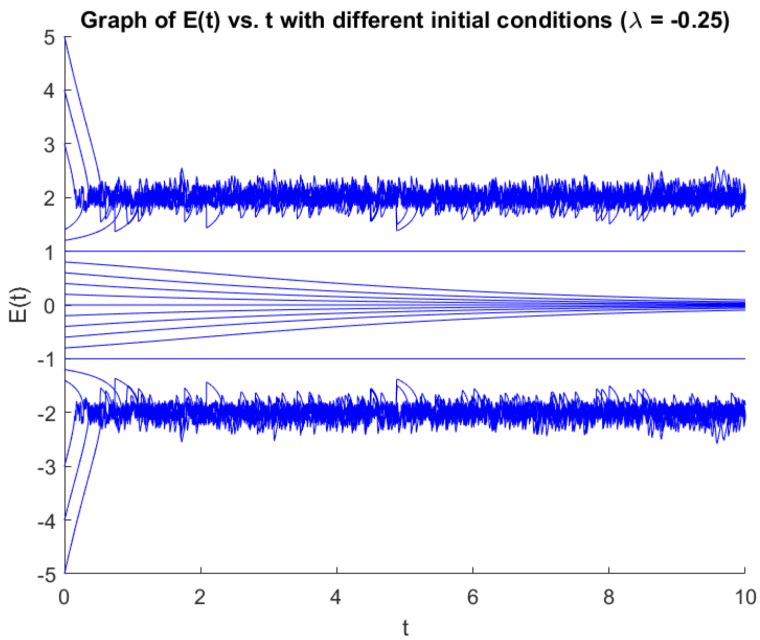
$$\begin{aligned} \dot{E} &= \kappa(P - E) & \dot{D} &= \gamma_2(\lambda + 1 - D - \lambda EP) \\ \dot{E} &= \kappa(ED - E) & \dot{D} &= \gamma_2(\lambda + 1 - D - \lambda E^2 D). \end{aligned}$$



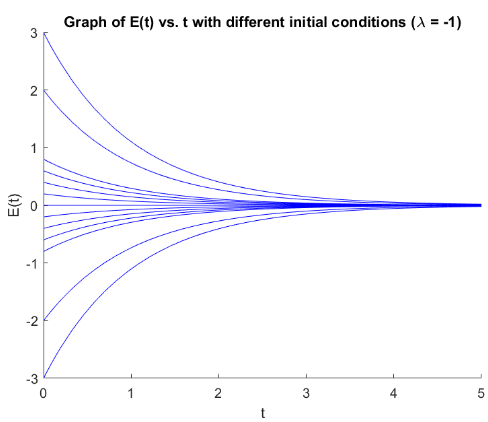
(a) $\lambda = 1$



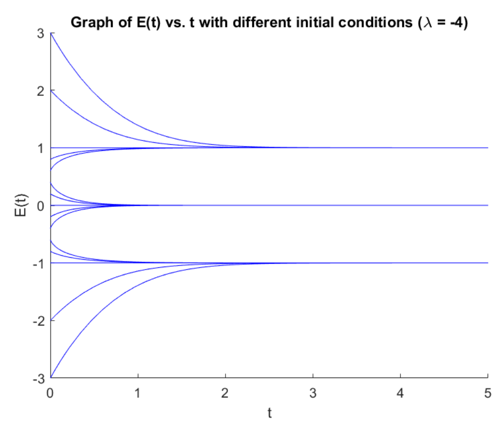
(b) $\lambda = 0$



(c) $\lambda = -0.25$



(d) $\lambda = -1$



(e) $\lambda = -4$

Figure 1: Plots of $E(t)$ with different initial conditions with $\kappa = 1$ and various λ: (a) $\lambda = 1$, (b) $\lambda = 0$, (c) $\lambda = -0.25$, (d) $\lambda = -1$, (e) $\lambda = -4$.

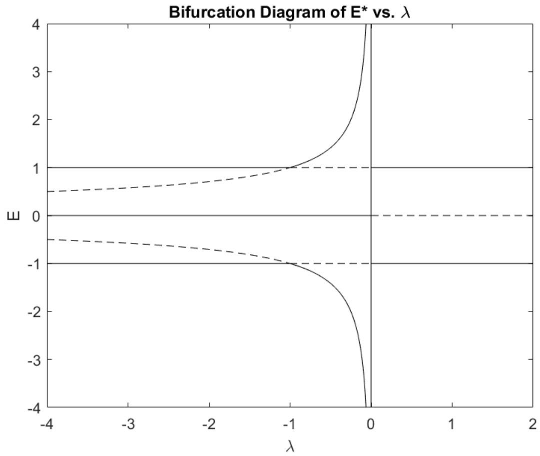


Figure 2: The bifurcation diagram of E^* vs. λ with $\kappa = 1$.

Next, we try to find the fixed points (E^*, D^*) by first finding the E- and D-nullclines.

$$\dot{E} = \kappa(ED - E) = 0 \qquad\qquad \dot{D} = \gamma_2(\lambda + 1 - D - \lambda E^2 D) = 0$$

$$E = 0, \quad D = 1 \qquad\qquad D = \frac{\lambda + 1}{1 + \lambda E^2}.$$

$$E = 0, \quad D = \frac{\lambda + 1}{1} = \lambda + 1 \quad \Rightarrow \quad (E^*, D^*) = (0, \lambda + 1).$$

$$D = 1, \quad 1 = \frac{\lambda + 1}{1 + \lambda E^2} \quad \Rightarrow \quad (E^*, D^*) = (-1, 1), \quad (E^*, D^*) = (1, 1).$$

E-nullcline $E = 0$ always intersects D-nullcline $D = \frac{\lambda+1}{1+\lambda E^2}$ at point $(0, \lambda + 1)$, which is a fixed point for the system, but the number of intersections between E-nullcline $D = 1$ and D-nullcline $D = \frac{\lambda+1}{1+\lambda E^2}$ depends on the parameter λ. Infinitely many intersection occurs when $\lambda = 0$, the intersections form a line of fixed points on $D = 1$. When $\lambda \neq 0$, there are two distinct intersections at $(-1, 1)$ and $(1, 1)$, suggesting two fixed points at these two locations.
The Jacobian matrix (E^*, D^*) is

$$A = \begin{pmatrix} \frac{\partial f}{\partial E} & \frac{\partial f}{\partial D} \\ \frac{\partial g}{\partial E} & \frac{\partial g}{\partial D} \end{pmatrix}\Bigg|_{(E^*,D^*)} = \begin{pmatrix} \kappa(D-1) & \kappa E \\ -2\gamma_2\lambda ED & -\gamma_2(1 + \lambda E^2) \end{pmatrix}\Bigg|_{(E^*,D^*)}$$

where $f(E, D) = \dot{E}$ and $g(E, D) = \dot{D}$.
At fixed point $(E^*, D^*) = (0, \lambda + 1)$,

$$A = \begin{pmatrix} \kappa\lambda & 0 \\ 0 & -\gamma_2 \end{pmatrix},$$

$\det(A) = -\gamma_2\kappa\lambda$, $\mathrm{Tr}(A) = \kappa\lambda - \gamma_2$, $\Delta = (\kappa\lambda - \gamma_2)^2 + 4\gamma_2\kappa\lambda = (\kappa\lambda + \gamma_2)^2 \geq 0$.
If $\lambda > 0$, $\det(A) < 0$, $(0, \lambda + 1)$ is a saddle point. If $\lambda = 0$, $\det(A) = 0$, $\mathrm{Tr}(A) < 0$, $(0, \lambda + 1)$ is on a line of stable fixed points, and that line is $D = 1$. If $\lambda < 0$, $\det(A) > 0$, $\mathrm{Tr}(A) < 0$, and because

$\Delta \geq 0$, $(0, \lambda+1)$ is either a sink ($\kappa\lambda + \gamma_2 \neq 0$, $\Delta > 0$) or a degenerate sink ($\kappa\lambda + \gamma_2 = 0$, $\Delta = 0$).
At fixed point $(E^*, D^*) = (-1, 1)$,

$$A = \begin{pmatrix} 0 & -\kappa \\ 2\gamma_2\lambda & -\gamma_2(1+\lambda) \end{pmatrix},$$

$\det(A) = 2\gamma_2\kappa\lambda$, $\mathrm{Tr}(A) = -\gamma_2(1+\lambda)$, $\Delta = \gamma_2^2(1+\lambda)^2 - 8\gamma_2\kappa\lambda = \gamma_2[\gamma_2(1+\lambda)^2 - 8\kappa\lambda]$.
If $\lambda < 0$, $\det(A) < 0$, $(-1, 1)$ is a saddle point. If $\lambda = 0$, $\det(A) = 0$, $\mathrm{Tr}(A) < 0$, $(-1, 1)$ is on a line of stable fixed points, and that line is $D = 1$. If $\lambda > 0$, $\det(A) > 0$, $\mathrm{Tr}(A) < 0$, $(-1, 1)$ is either a sink, a degenerate sink, or a spiral sink depending on the combination of the values of λ, κ, and γ_2. We will explain the exact conditions for getting different types of sinks after we walk through the fixed point $(1, 1)$.
At fixed point $(E^*, D^*) = (1, 1)$,

$$A = \begin{pmatrix} 0 & \kappa \\ -2\gamma_2\lambda & -\gamma_2(1+\lambda) \end{pmatrix},$$

$\det(A) = 2\gamma_2\kappa\lambda$, $\mathrm{Tr}(A) = -\gamma_2(1+\lambda)$, $\Delta = \gamma_2^2(1+\lambda)^2 - 8\gamma_2\kappa\lambda = \gamma_2[\gamma_2(1+\lambda)^2 - 8\kappa\lambda]$.
The $\det(A)$, $\mathrm{Tr}(A)$, and Δ for the fixed point $(1, 1)$ are the same as the ones for $(-1, 1)$, suggesting these two fixed points are the same type of fixed points regardless of the parameters. Thus, $(1, 1)$ is also a saddle point if $\lambda < 0$, a point on a line of stable fixed points if $\lambda = 0$, and sink, either normal, degenerate, or spiral sinks if $\lambda > 0$. We will analyze the conditions for getting different types of sinks for the two fixed points together with $\lambda > 0$.
Since the sign of Δ decides the types of sinks, we start by investigating the conditions when $\Delta = 0$,

$$\begin{aligned} \Delta = \gamma_2[\gamma_2(1+\lambda)^2 - 8\kappa\lambda] = 0 \\ \gamma_2\lambda^2 + 2(\gamma_2 - 4\kappa)\lambda + \gamma_2 = 0. \end{aligned}$$

By the quadratic formula, the solutions are

$$\begin{aligned} \lambda &= \frac{-2(\gamma_2 - 4\kappa) \pm \sqrt{2^2(\gamma_2 - 4\kappa)^2 - 4\gamma_2^2}}{2\gamma_2} \\ &= \frac{4\kappa - \gamma_2 \pm 2\sqrt{2\kappa(2\kappa - \gamma_2)}}{\gamma_2}. \end{aligned}$$

If $2\kappa < \gamma_2$, there is no real solution for $\Delta = 0$, suggesting $\Delta > 0$ regardless of values of λ, so both the fixed points $(-1, 1)$ and $(1, 1)$ are sinks. If $2\kappa = \gamma_2$, there is one real solution for $\Delta = 0$, suggesting $\Delta > 0$ most of the time, and $\Delta = 0$ when $\lambda = 1$. The fixed points $(-1, 1)$ and $(1, 1)$ are sinks if $\lambda \neq 1$ or degenerate sinks if $\lambda = 1$.
The focus is $2\kappa > \gamma_2$, which is consistent with the case $\gamma_1 \gg \kappa \geq \gamma_2$. Under this condition, there are two real solutions for $\Delta = 0$, and the fixed points $(-1, 1)$ and $(1, 1)$ can be sinks, degenerate sinks,

or spiral sinks. For simper notations for analyze, we denote

$$\lambda_1 = \frac{4\kappa - \gamma_2 - 2\sqrt{2\kappa(2\kappa - \gamma_2)}}{\gamma_2}, \quad \lambda_2 = \frac{4\kappa - \gamma_2 + 2\sqrt{2\kappa(2\kappa - \gamma_2)}}{\gamma_2}.$$

λ_1 is always positive ($\lambda_1 > 0$), and this can be checked by setting the expression of $\lambda_1 > 0$. Ultimately, we have $\gamma_2^2 > 0$, which supports the fact $\lambda_1 > 0$. Because λ_2 is larger than λ_1, the relationship is $0 < \lambda_1 < \lambda_2$.

Under the condition $2\kappa > \gamma_2$, for any λ that has value between λ_1 and λ_2 ($\lambda_1 < \lambda < \lambda_2$), $\Delta < 0$, and the fixed points $(-1, 1)$ and $(1, 1)$ are spiral sinks. For any λ that has value equals to either λ_1 or λ_2 ($\lambda = \lambda_1$ or $\lambda = \lambda_2$), $\Delta = 0$, and the two fixed points are degenerate sinks. For any positive λ that has value outside the range $[\lambda_1, \lambda_2]$ ($0 < \lambda < \lambda_1$ or $\lambda > \lambda_2$), $\Delta > 0$, and the two fixed points are sinks.

The above analysis applies in most cases. However, because the system is not linear, cases on the boundary such as degenerate sink or line of fixed points may not actually appear. Thus, we plot the system with different combinations of parameters in Figure 3 to check our analysis.

Figure 3a and Figure 3b have the same κ and negative λ but different γ_2. Both the fixed points $(-1, 1)$ and $(1, 1)$ in these two plots are saddle points, and $(0, 0)$ is the sink. From the analysis, $(0, 0)$ is the common sink in Figure 3a and degenerate sink in Figure 3b. The trajectories in both plots either converge toward $(0, 0)$ or diverge to the infinity in the first and second quadrant. Although the simulations show divergences to either the first or second quadrant with certain set of parameters, we do not expect the lasers to exhibit such behavior in reality. We never emit a laser that is continuously increasing in strength. Thus, the model breaks and does not predict well on the behaviors of the laser once the trajectories start to diverge away.

Figure 3c have $\lambda = 0$. All trajectories are moving towards the line of fixed points $D = 1$ while curving outward away from $E = 0$ with $D > 1$ and curving inward to $E = 0$ with $D < 1$.

We plot Figure 3d, Figure 3e, and Figure 3f with different positive λ and fixed κ and γ_2 to see if the types of sinks for $(-1, 1)$ and $(1, 1)$ are indeed different. The fixed point $(0, \lambda + 1)$ is a saddle point in all three plots, so all points on the line $E = 0$ converges to this fixed point. The types of sinks for $(-1, 1)$ and $(1, 1)$ are a bit hard to see from the plots. According to our analysis, Figure 3d should have common sinks, Figure 3e should have spiral sinks, and Figure 3f should have degenerate sinks. However, it looks like Figure 3d have degenerate sinks instead of common sinks.

3.3.3 Case 3

Finally, we have the case $\gamma_1 \approx \kappa \approx \gamma_2$. In this case, we are dealing with a 3D nonlinear system

$$\begin{aligned} \dot{E} &= \kappa(P - E) \\ \dot{P} &= \gamma_1(ED - P) \\ \dot{D} &= \gamma_2(\lambda + 1 - D - \lambda EP). \end{aligned}$$

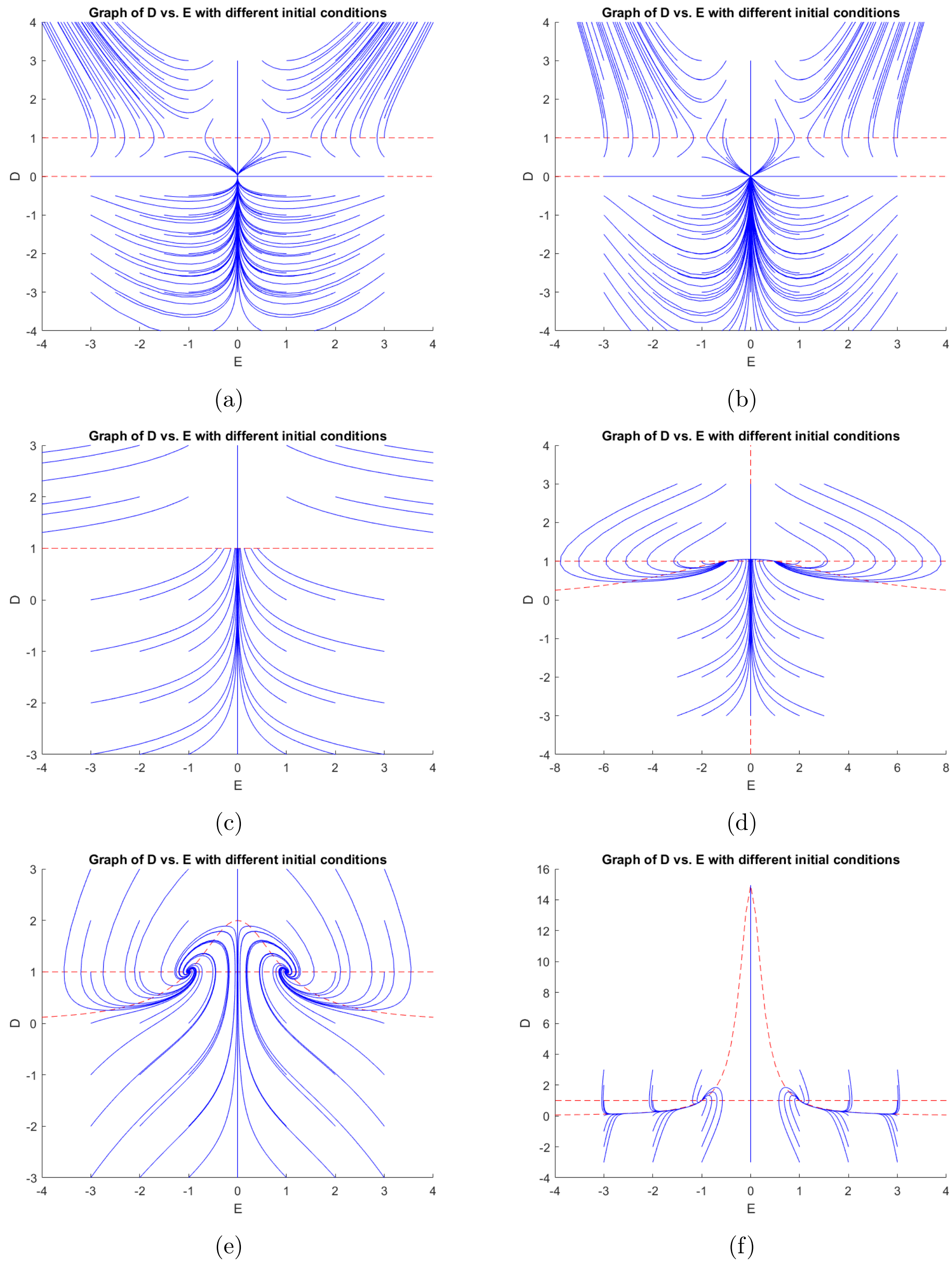


Figure 3: Plots of D vs. E with different initial conditions with various κ, γ_2 and λ: (a) $\kappa = 1$, $\gamma_2 = 0.5$, $\lambda = -1$; (b) $\kappa = 1$, $\gamma_2 = 1$, $\lambda = -1$; (c) $\kappa = 1$, $\gamma_2 = 0.5$, $\lambda = 0$; (d) $\kappa = 1$, $\gamma_2 = 0.5$, $\lambda = 0.05$ (e) $\kappa = 1$, $\gamma_2 = 0.1$, $\lambda = 1$, (f) $\kappa = 1$, $\gamma_2 = 0.5$, $\lambda = 7 + 4\sqrt{3}$. Red dashed lines are the nullclines.

First, we find the fixed points for the system.

$$\dot{E} = \kappa(P - E) = 0 \qquad \dot{P} = \gamma_1(ED - P) = 0 \qquad \dot{D} = \gamma_2(\lambda + 1 - D - \lambda EP) = 0$$
$$P = E \qquad P = ED \qquad D = \lambda + 1 - \lambda EP.$$

If $\lambda = 0$, there are infinitely many fixed points on the line $E = P$ on the plane $D = 1$. The fixed points are $(E, P, 1)$, where $E = P$.
If $\lambda \neq 0$, there are three fixed points: $(0, 0, \lambda + 1)$, $(1, 1, 1)$, and $(-1, -1, 1)$.
Next, we compute the 3×3 Jacobian matrix,

$$A = \left.\begin{pmatrix} \frac{\partial f}{\partial E} & \frac{\partial f}{\partial P} & \frac{\partial f}{\partial D} \\ \frac{\partial g}{\partial E} & \frac{\partial g}{\partial P} & \frac{\partial g}{\partial D} \\ \frac{\partial h}{\partial E} & \frac{\partial h}{\partial P} & \frac{\partial h}{\partial D} \end{pmatrix}\right|_{(E^*,P^*,D^*)} = \left.\begin{pmatrix} -\kappa & \kappa & 0 \\ \gamma_1 D & -\gamma_1 & \gamma_1 E \\ -\gamma_2\lambda P & -\gamma_2\lambda E & -\gamma_2 \end{pmatrix}\right|_{(E^*,D^*)}$$

where $f(E, P, D) = \dot{E}$, $g(E, P, D) = \dot{P}$, and $h(E, P, D) = \dot{D}$. Because the Jacobian matrix is 3×3, we could no longer use $\det(A)$ or $\mathrm{Tr}(A)$ to determine the stability of the fixed points as we did in Case 2. Rather, we pick certain sets of parameters at the fixed points, find the eigenvalues, and use the eigenvalues to determine the stability of the fixed points.
For the line of fixed points $(E, P, 1)$, where $E = P$, with $\lambda = 0$, assuming $E = P = a$, $a \in \mathbb{R}$,

$$A = \begin{pmatrix} -\kappa & \kappa & 0 \\ \gamma_1 & -\gamma_1 & \gamma_1 a \\ 0 & 0 & -\gamma_2 \end{pmatrix}.$$

We set $\kappa = \gamma_1 = \gamma_2 = 1$, $a = 1$, so the Jacobian matrix becomes

$$A = \begin{pmatrix} -1 & 1 & 0 \\ 1 & -1 & 1 \\ 0 & 0 & -1 \end{pmatrix}$$

with eigenvalues $\lambda_1 = 0$, $\lambda_2 = -1$, $\lambda_3 = -2$, and corresponding eigenvectors $\mathbf{v}_1 = (1, 1, 0)^T$, $\mathbf{v}_2 = (-2, 0, 1)^T$, $\mathbf{v}_3 = (-1, 1, 0)^T$. This suggests that the fixed point $(1, 1, 1)$ on a line of fixed points is a stable fixed points, and it is stable but not attracting on line spanned by $\mathbf{v}_1$, stable and attracting in space spanned by $\mathbf{v}_2$ and $\mathbf{v}_3$. Due to the non-linearity of the system, behaviors deviate if the points get away from the fixed points.
We do similar things to the other fixed points on the line of fixed points, and all other fixed points on the line have the same eigenvalues and varying eigenvectors for $\mathbf{v}_2$. This suggests a line of stable fixed points on line $(E, P, 1)$ with $E = P$. Points on the line are not attracting to the initial conditions that land on the line, but they are attracting to the initial conditions that fall outside of the line.

At fixed point $(E^*, P^*, D^*) = (0, 0, \lambda + 1)$, $\lambda \neq 0$,

$$A = \begin{pmatrix} -\kappa & \kappa & 0 \\ \gamma_1(\lambda+1) & -\gamma_1 & 0 \\ 0 & 0 & -\gamma_2 \end{pmatrix}.$$

We set $\kappa = \gamma_1 = \gamma_2 = 1$ and $\lambda = 15 > 0$, so the Jacobian matrix becomes

$$A = \begin{pmatrix} -1 & 1 & 0 \\ 16 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

with eigenvalues $\lambda_1 = -1$, $\lambda_2 = 3$, $\lambda_3 = -5$, and corresponding eigenvectors $\mathbf{v}_1 = (0, 0, 1)^T$, $\mathbf{v}_2 = (1, 4, 0)^T$, $\mathbf{v}_3 = (-1, 4, 0)^T$. Therefore, the fixed point $(0, 0, 16)$ is attracting in two directions $\mathbf{v}_1$ and $\mathbf{v}_3$ but repelling in one direction $\mathbf{v}_2$. Thus, we may say this fixed point is a saddle point. In fact, when $\lambda > 0$, the fixed point $(0, 0, \lambda + 1)$ is always a saddle point with two negative and one positive eigenvalues.

We also set $\kappa = \gamma_1 = \gamma_2 = 1$ and $\lambda = -0.51 < 0$, so the resulting Jacobian matrix has eigenvalues $\lambda_1 = -1$, $\lambda_2 = -0.3$, $\lambda_3 = -1.7$, and corresponding eigenvectors $\mathbf{v}_1 = (0, 0, 1)^T$, $\mathbf{v}_2 = (10, 7, 0)^T$, $\mathbf{v}_3 = (-10, 7, 0)^T$. Thus, the fixed point $(0, 0, 0.49)$ is attracting in all three directions. Thus, we say this fixed point is a sink. In fact, when $0 > \lambda > -1$, the fixed point $(0, 0, \lambda + 1)$ is always a sink with three negative eigenvalues.

We then set $\kappa = \gamma_1 = \gamma_2 = 1$ and $\lambda = -5 < -1$, and the resulting Jacobian matrix has eigenvalues $\lambda_1 = -1$, $\lambda_2 = -1+2i$, $\lambda_3 = -1-2i$ and corresponding eigenvectors $\mathbf{v}_1 = (0, 0, 1)^T$, $\mathbf{v}_2 = (-i, 2, 0)^T$, $\mathbf{v}_3 = (i, 2, 0)^T$. Thus, the fixed point $(0, 0, -4)$ is a sink in line spanned by $\mathbf{v}_1$ and a spiral sink in space spanned by $\mathbf{v}_2$ and $\mathbf{v}_3$. The trajectories that are close enough to this fixed point may converge towards it while keep spiraling. In fact, when $\lambda < -1$, the fixed point $(0, 0, \lambda + 1)$ is always a spiral sink with one real and two complex eigenvalues that have the same negative real parts.

At fixed point $(E^*, P^*, D^*) = (1, 1, 1)$, $\lambda \neq 0$,

$$A = \begin{pmatrix} -\kappa & \kappa & 0 \\ \gamma_1 & -\gamma_1 & \gamma_1 \\ -\gamma_2\lambda & -\gamma_2\lambda & -\gamma_2 \end{pmatrix}.$$

We set $\kappa = \gamma_1 = \gamma_2 = 1$ and $\lambda = 15 > 0$, so the resulting Jacobian matrix has eigenvalues $\lambda_1 = -1$, $\lambda_2 = -0.5 + i\frac{\sqrt{59}}{2}$, $\lambda_3 = -0.5 - i\frac{\sqrt{59}}{2}$.

When $\kappa = \gamma_1 = \gamma_2 = 1$ and $\lambda = 15$, the fixed point $(1, 1, 1)$ is a sink spanned by $\mathbf{v}_1$ and a spiral sink in space spanned by $\mathbf{v}_2$ and $\mathbf{v}_3$. The trajectories that are close enough to this fixed point may converge towards it while keep spiraling. In general, when $\lambda > b$, where b a point (threshold) between 0 and 1, the fixed point $(1, 1, 1)$ is a spiral sink with one real and two complex eigenvalues, and all eigenvalues have negative real parts.

If $b > \lambda > 0$, the fixed point $(1, 1, 1)$ is a sink with three negative eigenvalues.

We then set $\kappa = \gamma_1 = \gamma_2 = 1$ and $\lambda = -0.51 < 0$, the resulting Jacobian matrix has eigenvalues $\lambda_1 = -2$, $\lambda_2 = \frac{-5+2\sqrt{19}}{10}$, $\lambda_3 = \frac{-5-2\sqrt{19}}{10}$. Because this fixed point has two negative and one positive eigenvalues, this fixed point is a saddle point. Moreover, it applies to conditions with other values of λ that satisfy $\lambda < 0$.

At fixed point $(E^*, P^*, D^*) = (-1, -1, 1)$, $\lambda \neq 0$,

$$A = \begin{pmatrix} -\kappa & \kappa & 0 \\ \gamma_1 & -\gamma_1 & -\gamma_1 \\ \gamma_2\lambda & \gamma_2\lambda & -\gamma_2 \end{pmatrix}.$$

By similar process, we fix $\kappa = \gamma_1 = \gamma_2 = 1$, tune λ, find the eigenvalues, and determine the stability.

When $\lambda > b$, the fixed point $(-1, -1, 1)$ is a spiral sink with one real and two complex eigenvalues, and all eigenvalues have negative real parts.

When $b > \lambda > 0$, the fixed point $(-1, -1, 1)$ is a sink with three negative eigenvalues.

When $\lambda < 0$, the fixed point $(-1, -1, 1)$ is a saddle point with two negative and one positive eigenvalues.

In order to make the above analysis more straightforward, we have some 3D plots in Figure 4 that help us to understand the behavior of the system with different conditions. All plots have fixed parameters $\kappa = \gamma_1 = \gamma_2 = 1$, the only parameter that varies is λ. We will progress our explanation with decreasing λ from positive to negative.

We start with plot $\lambda = 15$ in Figure 4a. From our earlier analysis, the two fixed points at $(1, 1, 1)$ and $(-1, -1, 1)$ are the spiral sinks, and the fixed point at $(0, 0, \lambda + 1)$ is the saddle point. The plot supports our analysis by showing trajectories converging to one of the two points $(1, 1, 1)$ and $(-1, -1, 1)$ and forming two spirals. Moreover, trajectories with initial conditions on line $(0, 0, D)$, where $D \in \mathbb{R}$, move towards $(0, 0, \lambda + 1)$, but trajectories not on that line diverge away to either one of the two spiral sinks.

With $\lambda = 0$, the plot in Figure 4b shows a line of stable fixed points, where the line satisfies $E = P$ and $D = 1$. The plot not only supports our analysis but also shows some behaviors of the system away from the line of fixed points.

Both plots in Figure 4c and Figure 4d have $\lambda < 0$. The fixed $(1, 1, 1)$ and $(-1, -1, 1)$ are saddle points in both plots, but the types of fixed point $(0, 0, \lambda + 1)$ are different. $(0, 0, \lambda + 1)$ is a sink in Figure 4c but a spiral sink in Figure 4d.

3.3.4 Case 3 Interpretation

Let us interpret the case where all three equations are considered. Reflecting upon the physical scenario, we can find that Figure 4a corresponds to the laser process where the threshold is met since there are nonzero E fixed points. Figure 4c and Figure 4d represent a scenario where energy is extracted from the system and E and P are brought to rest. The saddle points are likely products of the mathematical model as the system would move around those points and, realistically, any initial conditions at a saddle node would get kicked off due to noise and head towards the sink. The

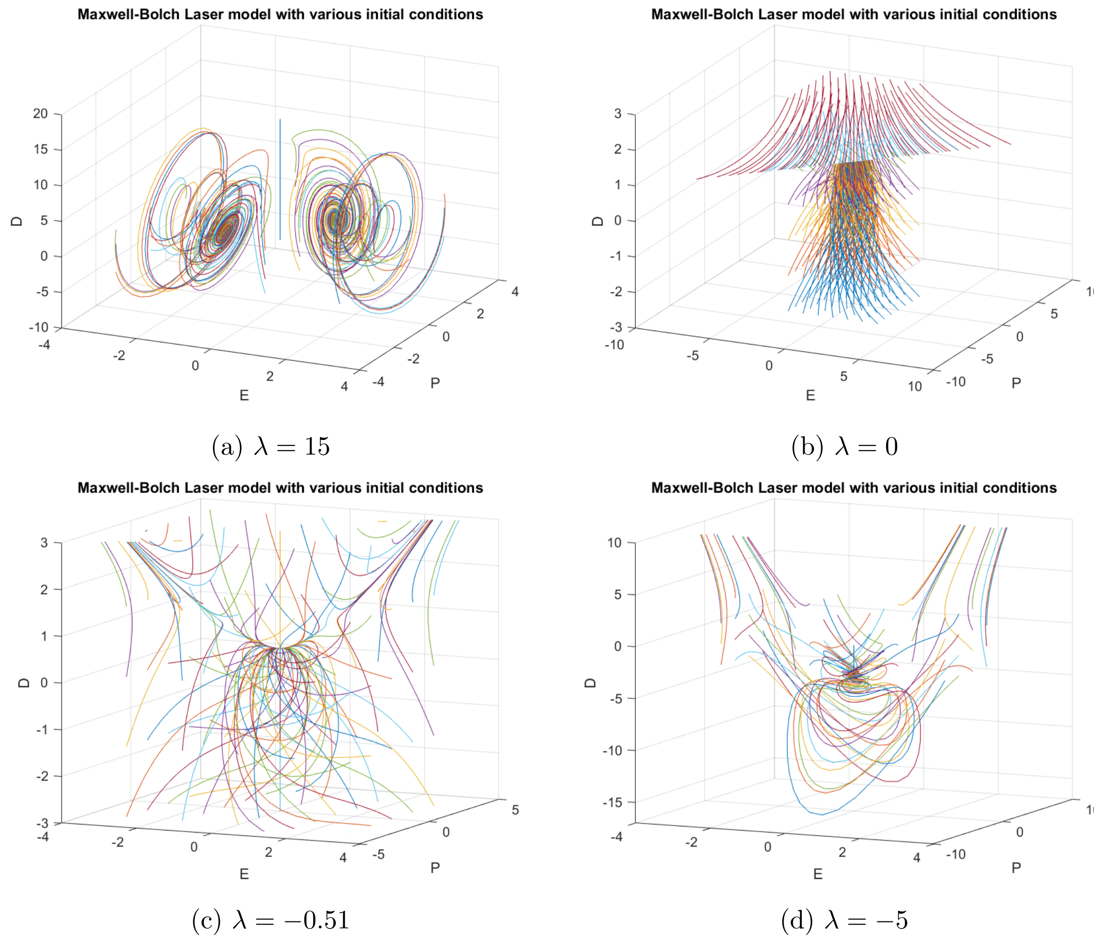


Figure 4: Plots of D vs. P vs. E with various initial conditions, fixed parameters $\kappa = \gamma_1 = \gamma_2 = 1$, and varying parameter λ: (a) $\lambda = 15$; (b) $\lambda = 0$; (c) $\lambda = -0.51$; (d) $\lambda = -5$.

case where $\lambda = 0$ in Figure 4b is a bit trickier to explain. As we've noted above, points that fall outside of the line attract towards the line. Thus, we can imagine that if there is no pumping, the strength of the electric field and atom polarization will eventually become equal and 'balance' each other out while having a constant value for population inversion.

4 Summary and Discussion

In summary, most cases for the Maxwell-Bloch model tend to have three fixed points where there may either be two non-zero sinks that suggest laser emission or one zero sink node that suggests the laser dying out in the long term. However, the model breaks sometimes with specific set of parameters, suggesting it may not predict well with certain conditions. This is an issue that we predicted in the introduction by consider the similarity to a Lorenz system, which introduces chaotic behavior. In general, we would expect our system to either die out or have stable, coherent emission in the long term and create a laser in the physical world. Such behavior is common between the 2-D and 3-D models, in which a sufficiently large pumping parameter results in a sustained laser process and negative pumping intuitively leads to the system going to zero. Setting their respective pumping parameters to zero leads to questionable results that differ between the two models. Due to the supposedly higher accuracy of Maxwell-Bloch, the 3-D model may be more realistic in showing that the system would find a balance rather than go to zero. Results from the models can be verified with experiments by tuning and manipulating the setting of the laser emission instrument and observing the behavior of lasers in the long term. The actual experiments are especially useful for checking the behavior of lasers in cases where our model fails to predict. Some possible improvements of the model may include changing the assumptions for the model, introducing one more term to the equations that relates to the loss of energy/accounts for noise, or adding more equations to the system. Moreover, some other literatures have elaborated upon the chaotic behavior of the lasers under certain conditions[10]. We have not found this chaotic behavior yet in our models, so the potential future direction may be discovering the conditions for chaotic behavior and connecting it to the Lorenz equations. Another potential future direction may be finding an analytic solution for the pumping threshold for the 3-D system. It would also be possible to tweak the terms in the model to better fit the physical scenario and make the model more robust but this would require a deeper understanding of quantum electrodynamics.

References

[1] Carlos J. Bustamante, Yann R. Chemla, Shixin Liu, Michelle D. Wang, *Optical tweezers in single-molecule biophysics*, Springer Nature, (2021)

[2] Gernot Salzer, *Example: Poincaré, Classification of Phase Portraits*, TEXample.net, (2018)

[3] Max Planck, *Zur Theorie des Gesetzes der Energieverteilung im Normalspectrum*, Verhandlungen der Deutschen Physikalischen Gesellschaft, (1900)

[4] Daniel E. Bath, John R. Stowers, Dorothea Hormann, et al., *FlyMAD: rapid thermogenetic control of neuronal activity in freely walking Drosophila*, Nature Methods, (2014)

[5] Steven H. Strogatz, *Nonlinear Dynamics and Chaos*, CRC Press, (2000)

[6] Hermann Haken, *Synergetics: An Introduction*, Springer, (1983)

[7] Peter W. Milonni, Joseph H. Eberly, *Lasers*, Wiley-Interscience, (1988)

[8] C.O. Weiss, R. Vilaseca, *Dynamics of Lasers*, Wiley-VCH, (1991)

[9] Hermann Haken, *Analogy between higher instabilities in fluids and lasers*, Elsevier, (1975)

[10] Rogelio Ospina, F. Mesa, J. Granada, *Analysis of a strange attractor of the laser type by means of the Maxwell-Bloch equations using the Lorentz equations*, IOP Publishing, (2021)

[11] Fortunato Tito Arecchi, Riccardo Meucci, *Chaos in lasers*, Scholarpedia, (2008)